CONDITIONAL EXPECTATIONS, TRACES, ANGLES BETWEEN SPACES AND REPRESENTATIONS OF THE HECKE ALGEBRAS

FLORIN RĂDULESCU*

DIPARTIMENTO DI MATEMATICA
UNIVERSITA DEGLI STUDI DI ROMA "TOR VERGATA"

ABSTRACT. In this paper we extend the results in [Ra] on the representation of the Hecke algebra, determined by the matrix coefficients of a discrete series, projective, unitary representation, of the ambient group to a more general, vector valued case. This method could be used to analyze the traces of the Hecke operators.

We construct representations the Hecke algebra of a group G relative to an almost normal subgroup Γ in the ring (von Neumann) algebra of the group G tensor matrices. These representations are a lifting of Hecke operators to this larger algebra. By summing up the coefficients of the terms in the representation one obtains the classical Hecke operators.

These representations were used in the scalar case in [Ra], to find an alternative representation of the Hecke operators on Maass forms, and hence to reformulate the Ramanujan Petersson conjectures as a problem on the angle (see e.g. A. Connes's paper [Co] on the generalization of CKM matrix) between two subalgebras of the von Neumann algebra of the group G: the image of the representation of the Hecke algebra and the algebra of the almost normal subgroup.

INTRODUCTION

Let G be a countable discrete group and $\Gamma \subseteq G$ be an almost subgroup. Assume π is a unitary representation of G into the unitary group of a separable Hilbert space H.

We assume that $\pi|_\Gamma$ is a multiple of the left regular representation λ_Γ of Γ on $l^2(\Gamma)$. For simplicity, throughout the paper we assume that the groups Γ and G have infinite conjugacy classes, and hence the associated von Neumann

* Member of the Institute of Mathematics "S. Stoilow" of the Romanian Academy

algebras are factors and thus have a unique trace denoted by τ. The Murray von Neumann theory of dimension (see e.g. [Ta], [GHJ]) associates to this representation of the group Γ, a continuous dimension $t = \dim_{\pi(\Gamma)''} H = \dim_\Gamma H \in (0, \infty]$. Here, by $\pi(\Gamma)''$ we denote, as customary in von Neumann algebras, the von Neumann algebra geneated by $\pi(\Gamma) \subseteq B(H)$. In the paper [Ra] we treated the case $\dim_\Gamma H = 1$. (We are deeply indepted to H. Moscovici for suggesting to investigate the general case of arbitrary dimension). If t is an integer or ∞ on infinity, the hypothesis on the representation of G corresponds to the existence of a Hilbert subspace $L \subseteq H$, such that $H \cong l^2(\Gamma) \otimes L$, and $\pi = \lambda_\Gamma \otimes \mathrm{Id}_L$.

To this data we associate a representation S of the algebra of Hecke operators into the reduced von Neumann algebra of the right regular representation of G, $R(G)_t$ ([MvN]). (If t is an integer, the algebra $R(G)_t$ is $R(G) \otimes B(K)$, where K is any Hilbert space of dimension $\dim_{\mathbb{C}} K = t$), otherwise $R(G)_t = p(R(G) \otimes B(L))p$, where p is a projection in $\mathcal{L}(\Gamma) \otimes B(L)$ of trace $\tau(p) = t$, and L is an arbitrary infinite dimensional Hilbert space.

In the case $\dim_{\mathbb{C}} L = \infty$, the representation takes values into the formal ring of infinite series in G with coefficients in $B(L)$.

Let $\mathcal{H}_0 = \mathbb{C}(\Gamma \setminus G/\Gamma)$ be the $\mathbb{C}$ - algebra of double cosets. Let $\mathcal{H}$ be the reduced von Neumann - Hecke algebra of $\Gamma \subseteq G$, acting on $l^2(\Gamma/G)$ (the weak operator topology closure of $\mathcal{H}_0$ viewed as a subalgebra of $B(l^2(\Gamma/G))$, see [BC]).

We generalise the results in [Ra], by proving also in this more general case, that the angle (see Theorem 8) between the algebra $S(\mathcal{H}) \subseteq R(G)_t$ and the algebra $R(\Gamma)_t$ determines the Ramanujan - Petterson behaviour of the representation Ψ of the (algebraic) Hecke algebra $\mathcal{H}_0 = \mathbb{C}(\Gamma \setminus G/\Gamma)$ (the $\mathbb{C}$ - algebra of double cosets) on $\pi(\Gamma)' \subseteq B(H)$. Here G acts on $B(H)$ by $\mathrm{Ad}\pi(g), g \in G$. Then $\pi(\Gamma)'$ is the commutant algebra. By the Murray von Neumann dimension theory the commutant algebra is isomorphic to $R(G)_t$. As proved in ([Ra]) an essential ingredient to prove that the representations of the Hecke algebra may be used to construct Hecke operators on $\pi(\Gamma)'$ (which is then equivalent the problem of determining the action of Hecke operators on Maass forms), is the fact that the representation of the Hecke algebra may be extended naturally to an operator system containing the Hecke algebra (conform Definition 1).

The representation Ψ is simply the action of the Hecke algebra on Γ invariant elements in $B(H)$. The space of the Γ- invariant elements is exactly the algebra $\pi(\Gamma)'$.

If $\Gamma = PSL_2(\mathbb{Z})$, $G = PGL_2(\mathbb{Z}_{\frac{1}{p}})$, π is the restriction to G of a representation on the discrete series of $PGL_2(\mathbb{R})$ (could also be projective), then, through Berezin symbol map ([Be]), one recaptures from the representation Ψ the classical Hecke operators on Maass forms.

The explanation of the fact that we are able to recapture the action of classical Hecke operators on Γ invariant vectors is as follows: Starting with the representation S of the Hecke algebra into $p(R(G) \otimes B(L))p$ one defines the densely defined, $*$-algebra morphism, $\widetilde{\varepsilon} = \varepsilon \otimes \mathrm{Id}$ from $R(G) \otimes B(L)$ into $B(L)$ by letting ε be the character of $R(G)$ (that is $\varepsilon(\sum a_g \rho_g) = \sum a_g$).

If $S(\mathcal{H}_0)$ is contained in the domain of $(\varepsilon \otimes \mathrm{Id})$, then $\widetilde{\varepsilon} \circ S$ is a representation of S onto $B(L)$ (in fact $\varepsilon(p)L$). Identifying $\varepsilon(p)L$ with the Γ-invariant vectors in the representation π, we obtain the representation of the classical operators Hecke operators on Γ-invariant vectors.

In particular applying the above construction to the representation $\pi \otimes \overline{\pi}$ acting on $H \otimes \overline{H}$, one obtains classical Hecke operators on the Hilbert space of Γ - invariant vectors in $H \otimes \overline{H}$. This latest Hilbert space is canonically identified to the Hilbert space $L^2(\pi(\Gamma)', \tau)$ associated to the von Neumann algebra $\{\pi(\Gamma)\}'$.

We use the following construction of the Hilbert space of the associated Γ - invariant vectors. We assume the representation π, now denoted by π_0 acts on the Hilbert space H_0. To define a Hilbert space of Γ - invariant vectors we use a scale (as it is the case with the Petterson scalar product).

Thus, we assume that $H_0 \subseteq H$, and we assume that there exists π, a representation of G on H, such that π invariates H_0, and $\pi(g)|_{H_0} = \pi_0(g), g \in G$.

We assume $L \subseteq H$, is a subspace such that $\pi|_\Gamma \cong \lambda_\Gamma \otimes \mathrm{Id}_L$ (thus $H \cong l^2(\Gamma) \otimes L$) and define the Hilbert space of Γ-invariant vectors (as in the Petersson scalar product) by letting $H_0^\Gamma = H_0^\Gamma(\pi, L)$ be the subspace of densely defined forms, on H_0, such that $\langle v, v \rangle_{H_0^\Gamma} = \langle P_L v, v \rangle$ is finite (thus such that the densely defined map on H associating to a vector w in the domain the value $\langle P_L v, w \rangle = \langle v, P_L w \rangle$ extends to a continuous form on H).

Then $\widetilde{\varepsilon} \circ S$ is the classical Hecke operator representation on $H_0^\Gamma(\pi, L)$. (see Chapter 3) This, also allows to obtain another representation of the Hecke operators on Γ invariant vectors associated to the action of Γ on H_0, as operators acting on a subspace of L. These representation can be used to analyze the traces of the Hecke operators.

Finally, in the last chapter we determine a precise formula for the absolute value of the matrix coefficients of the 13-th element π_{13} in the discrete series of $PSL_2(\mathbb{R})$, when restricted to $G = PGL_2(\mathbb{Z}[\frac{1}{p}])$.

This proves that this coefficients are absolutely summable on cosets of $\Gamma = PSL_2(\mathbb{Z})$.

1. The representation of the Hecke algebra

In this chapter, we introduce a representation of the Hecke algebra of an almost normal subgroup Γ of a discrete group G, that is canonicaly associated to a projective, unitary representation π of the larger group G.

Assume G is a discrete countable group and $\Gamma \subseteq G$ an almost normal subgroup. Let $\pi : G \to B(H)$ be a projective, unitary representation of the group G into the unitary operators on the Hilbert space H.

We assume that $\pi|_\Gamma$ is multiple representation of the left regular representation of the group Γ (which as observed in [Ra]) implies that the associated Hecke algebra $\mathcal{H}_0 = \mathbb{C}(\Gamma \setminus G/\Gamma)$ is unimodular, i.e. $[\Gamma : \Gamma_\sigma] = [\Gamma : \Gamma_{\sigma^{-1}}]$, with $\Gamma_\sigma = \sigma\Gamma\sigma^{-1} \cap \Gamma$, for all $\sigma \in G$.

In addition throught this chapter we assume that there exists a Hilbert subspace L of H such that L is generating, and Γ - wandering for H (that is γL is orthogonal to L for all γ in $\Gamma \setminus \{e\}$, and the closure of the linear span $\overline{Sp(\bigcup_{\gamma\in\Gamma} \gamma L)}$ is equal to H). Here by e we denote the identity element of the group G.

Then H is identified to the Hilbert space $l^2(\Gamma) \otimes L$, and L is identified to $\{e\} \otimes L$, and the representation π is identified to the left regular representation λ_Γ of Γ on $l^2(\Gamma)$, tensor the identity of L.

In the sequel we denote the right regular representation of Γ (respectively G) by ρ_Γ (respectively ρ_G) on $l^2(\Gamma)$ (respectively $l^2(G)$). When no confusion is possible we simply denote this by ρ. By $\rho\widehat{(G) \otimes B}(L)$ we denote the algebra of formal series

$$\sum_{g\in G} \rho(g) \otimes A_g$$

with coefficients $A_g \in B(L)$, $g \in G$, that have the additional property that for every $l_1, l_2 \in L$, the coefficient

$$\langle (\sum_{g\in G} \rho(g) \otimes A_g)(\delta_e \otimes l_1), \delta_e \otimes l_2 \rangle$$

defined by

$$\sum_{g\in G} \rho(g) \langle A_g l_1, l_2 \rangle$$

is an element of $R(G)$, the von Neumann algebra of right, bounded convolutors on $l^2(G)$.

With the above definitions, we consider the following linear operator system ([Pi]):

Definition 1. By $\mathcal{K}_0^*\mathcal{K}_0$ we denote the tensor product $\mathbb{C}(G \setminus \Gamma) \underset{\mathcal{H}_0}{\otimes} \mathbb{C}(\Gamma \setminus G)$. Here $\mathcal{H}_0 = \mathbb{C}(\Gamma \setminus \Gamma / G)$ is the Hecke algebra of double cosets and $\mathcal{H}_0$ is embedded both in $\mathbb{C}(G \setminus \Gamma)$ and $\mathbb{C}(\Gamma/G)$, simply by writting a double coset as a sum of right, respectively left, cosets.

There is an obviously $*$ operation on $\mathcal{K}_0^*\mathcal{K}_0$ (whence the notation). Then $\mathcal{K}_0 = \mathbb{C}(\Gamma/G)$, and $\mathcal{K}_0^* = \mathbb{C}(G \setminus \Gamma)$ are canonically identified as subspaces of $\mathcal{K}_0^*\mathcal{K}_0$. We have a canonical bilinear map from $\mathcal{K}_0^* \times \mathcal{K}_0$ onto $\mathcal{K}_0^*\mathcal{K}_0$ which when restricted to the Hecke algebra $\mathcal{H}_0$ becomes the Hecke algebra multiplication.

As observed in ([Ra]), the operator system $\mathcal{K}_0^*\mathcal{K}_0$ is isomorphic to the linear span of cosets:

$$\mathbb{C}\{\sigma_1\Gamma\sigma_2 | \sigma_1, \sigma_2 \in G\} = \mathbb{C}\{(\sigma_1\Gamma\sigma_1^{-1})\sigma_2 | \sigma_1, \sigma_2 \in G\},$$

where the bilinear operation maps $(\sigma_1\Gamma, \Gamma\sigma_2)$ into $\sigma_1\Gamma\sigma_2, \sigma_1, \sigma_2 \in G$.

There is an obviously completion of this system, which is still a $*$ operator system, which we denote as $\mathcal{K}^*\mathcal{K}$. This operator system is, by definition, $l^2(G/\Gamma) \underset{\mathcal{H}}{\otimes} l^2(\Gamma \setminus G)$, where $\mathcal{H}$ is the reduced von Neumann algebra of the Hecke algebra, that is the weak closure of $\mathcal{H}_0$ acting on $l^2(\Gamma \setminus G)$ (respectively on $l^2(G/\Gamma)$ by right multiplication).

A priori, then is no canonical Hilbert structure on $\mathcal{K}^*\mathcal{K}$, but using the construction in this paper, we define at the end of this paper a canonical Hilbert structure on $\mathcal{K}\mathcal{K}^*$ that is compatible with the action of $\mathcal{H}$ (that is we have a scalar product on cosets

$$\langle \Gamma\sigma_1 \otimes \sigma_2\Gamma, \Gamma\sigma_3 \otimes \sigma_4\Gamma \rangle$$

that depends only on $\sigma_3^{-1}\Gamma\sigma_1$, $\sigma_2^{-1}\Gamma\sigma_3$) and thus by cyclically rotating the variables, the scalar product corresponds to a bilinear form on $\mathcal{K}^*\mathcal{K}$. (Note that in [Ra] we constructed also a scalar product, with positive values on the generators, such that the corresponding bilinear form is again positive definite).

With the above defintions we have the following.

Proposition 2. *Let $\Gamma \subseteq G, \pi, H, L$ and $\mathcal{K}_0^*\mathcal{K}_0$ as above. Then there exists a canonical representation*

$$S : \mathcal{K}_0^*\mathcal{K}_0 \to R(\widehat{G) \otimes B}(L)$$

that is a $*$ morphism in the sense that $S(k_1)^* S(k_2) = S(k_1^* k_2)$ where $k_1^* k_2$ is viewed as an element of $\mathcal{K}_0^*\mathcal{K}_0$, for all $k_1, k_2 \in \mathcal{K}_0$. By this assertion we require that the elements $S(k_1^*) = S(k_1)^*$, $S(k_2)$ may be multiplied in $R(\widehat{G) \otimes B}(L)$, for all $k_1, k_2 \in \mathcal{K}_0$.

By restriction to $\mathcal{H}_0$ we obtain a representation of the Hecke algebra. When L is of finite dimension, this representation may be extended to $\mathcal{K}^*\mathcal{K}$ (and hence to $\mathcal{H}$). Let P_L be the the projection from H onto L.

The formula for S is

$$S(A) = \sum_{\theta \in A} \rho(\theta^{-1}) \otimes P_L \pi(\theta) P_L$$

where A is any of the cosets $\sigma_1\Gamma, \Gamma\sigma_2$ or $\sigma_1\Gamma\sigma_2 = [\sigma_1\Gamma][\Gamma\sigma_2], \sigma_1, \sigma_2 \in G$.

Proof. In the summation we have to take $\rho(\theta^{-1}$, for $\theta \in A$, since the right regular representation has the property that $\rho(a)\rho(b) = \rho(ba)$, for $a, b \in R(G)$.

The multiplicativity of the map S follows from the fact that

$$\sum_{\gamma \in \Gamma} P_L \pi(\theta_1 \gamma) P_L \pi(\gamma^{-1}\theta_2) P_L = P_L \pi(\theta_1\theta_2) P_L.$$

This formula is a consequence of the fact that $\sum \pi(\gamma) P_L \pi(\gamma)^{-1} = \mathrm{Id}$ (this is also valid for projective representations). This is indeed the coefficient of $\rho(\theta_1\theta_2)$ in a product $S(\theta_1\Gamma)S(\Gamma\theta_2)$ for all $\theta_1, \theta_2 \in G$.

The fact that the matrix coefficients of S belong indeed to $R(G)$, is observed as follows. Fix l_1, l_2 in L.

Then

$$\sum_{\theta \in \Gamma\sigma} \rho(\theta^{-1}) \langle \pi(\theta) l_1, l_2 \rangle = \sum_{\gamma \in \Gamma} \rho((\gamma\sigma)^{-1}) \langle \pi(\gamma)\pi(\sigma) l_1, l_2 \rangle =$$

$$= \sum_{\gamma \in \Gamma} \rho(\sigma^{-1}) \rho(\gamma^{-1}) \langle \pi(\sigma) l_1, \pi(\gamma^{-1}) l_2 \rangle$$

(since $\pi(\gamma) l_2$ are orthogonal).

The fact that S may be be continued to $\mathcal{K}^*\mathcal{K}$ in the case that L is finite dimensional is proved as follows. Indeed in this case the matrix entries $\omega_{l_1,l_2} = \langle l_1, \cdot \rangle l_2$, $l_1, l_2 \in L$, are when composed with S, the matrix entries of a representation of $\mathcal{H}_0$, into $R(\widehat{G) \otimes B}(L)$, which is however trace

preserving and thus extendable by continuity to $\mathcal{H}$. In particular, if L is finite dimensional, then for all $\sigma \in G$, we have that $S(\Gamma\sigma)$, $S(\sigma\Gamma)$ belong to $R(G) \otimes B(L)$. $\square$

Observation 3. We define formally

$$E_{\pi(\Gamma)'}(A) = \sum_{\gamma \in \Gamma} \pi(\gamma) A \pi(\gamma)^{-1},$$

for A in $B(H)$. If the above sum is so convergent, then this sum represents the conditional expectation onto $\pi(\Gamma)'$.

Moreover if $A \in \mathcal{C}_1(H)$ (i.e. A is a trace class operator), then the sum from the statement is simply the restriction of the state $\mathrm{tr}(A\cdot)$ to $\pi(\Gamma)'$. (Here the linear functional tr is the trace on the trace class operators $\mathcal{C}_1(H)$).

In particular if AP_L belongs to $\mathcal{C}_1(H)$ (or even $\mathcal{C}_2(H)$) then $E_{\pi(\Gamma)'}(AP_L)$ is clearly equal to $\sum\limits_{\gamma \in \Gamma} \rho(\gamma^{-1}) \otimes P_L \pi(\gamma) A P_L$.

In particular, the preceding formula for

$$S^{\sigma\Gamma} = \sum_{\gamma} \rho((\sigma\gamma)^{-1}) \otimes P_L \pi(\sigma) \pi(\gamma) P_L$$

might be interpreted as $\rho(\sigma^{-1}) E_{\pi(\Gamma)'}(\pi(\sigma) P_L)$.

Let ε be the densely defined character on $R(G)$ defined as the sum of coefficients. Then we define $\widetilde{\varepsilon} = \varepsilon \otimes \mathrm{Id}$ on $R(\widehat{G) \otimes B}(L)$ (densely defined) with values in $B(L)$ by the formula

$$\widetilde{\varepsilon}(\sum \rho(g) \otimes A_g) = \sum_g A_g$$

(for $x = \sum \rho(g) \otimes A_g \in R(\widehat{G) \otimes B}(L)$, the condition for x to be in the domain of $\widetilde{\varepsilon}$ is that the sum $\sum\limits_g A_g$ be no-convergent).

Then $\widetilde{\varepsilon} \circ S$, if $S(\mathcal{H}_0) \subseteq \mathrm{Dom}\, \widetilde{\varepsilon}$, is a representation of $\mathcal{H}_0$ into $B(L)$.

Observation 4. Assume that $(\mathcal{X}, \nu)$ is an infinite measure space and $H = L^2(\mathcal{X}, \nu)$.

Assume that G acts by measure preserving transformation on $\mathcal{X}$ and assume that F is a fundamental domain for Γ in $\mathcal{X}$. Let $\pi = \pi_{\mathrm{Koop}}$ be the Koopman representation (see [Ke]) of G on $L^2(\mathcal{X}, \nu)$. Let σ be any element

of G. Let χ_F be the characteristic function of F (viewed as a projection in $B(L^2(\mathcal{X},\nu))$). Then

$$S(\Gamma\sigma\Gamma)=\sum_{\theta\in\Gamma\sigma\Gamma}\rho(\theta^{-1})\otimes\chi_F\pi_{\mathrm{Koop}}(\theta)\chi_F,\sigma\in G.$$

Then we have that

$$\widetilde{\varepsilon}(S(\Gamma\sigma\Gamma))=\sum\chi_F\pi_{\mathrm{Koop}}(\theta)\chi_F,\sigma\in G.$$

This is exactly the classical Hecke operator associated to the double coset $\Gamma\sigma\Gamma$ acting on the L^2 space associated to the fundamental domain.

Observation 5. In the previous more general setting, we identify the Γ- wandering subspace L of H, with a subspace of Γ invariant vectors in H (here by Γ - invariant vectors we understand densely defined, Γ - invariant functionals on H).

Thus to every vector $l\in L$, we associate the Γ - invariant vector $\sum\limits_{\gamma\in\Gamma}\pi(\gamma)l$.

Then clearly

$$\widetilde{\varepsilon}(S(\Gamma\sigma\Gamma))l=\widetilde{\varepsilon}(S(\Gamma\sigma\Gamma))(P_L\sum_{\gamma\in\Gamma}\pi(\gamma)l)=\sum_{\theta\in\Gamma\sigma\Gamma}P_L\pi(\theta)l,l\in L,$$

This is the further equal to

$$P_L(\sum\pi(\sigma)\pi(s_i)(\sum_{\gamma\in\Gamma}\pi(\gamma)l))$$

,

and this is exactly the action of the classical Hecke operator associated to the double coset $\Gamma\sigma\Gamma$ on $\sum\pi(\gamma)l$ (which is $\sum\pi(\sigma s_i)\sum\limits_{\gamma\in\Gamma}\pi(\gamma)l=\sum\limits_{\gamma\in\Gamma}\pi(\gamma)(\sum\limits_{v_j}\pi(\sigma v_j)l)$).

Here s_i,v_j; are cosets representatives so that $\Gamma\sigma\Gamma=\cup s_i\sigma\Gamma=\cup\Gamma\sigma v_j$). In particular we obtain an extension from classical Hecke operators, which are a representation of $\mathcal{H}_0$ to a representation of $\mathcal{K}_o^*\mathcal{K}_0$.

One example, in which $\widetilde{\varepsilon}\circ S$ is well defined, is the case of a tensor product of two representations as above.

Proposition 6. *Let $\pi_i,G,\Gamma,L_i,H_i,i=1,2$ be as above and let $\pi:G\to H_1\otimes\overline{H}_2$ be the diagonal representation. We identify $H_1\otimes\overline{H}_2$ with $\mathcal{C}_2(H_1,H_2)$, the Schatten von Neumann class of operators from H_1 into H_2. In this identification, the representation π is the adjoint representation, mapping $X\in\mathcal{C}_2(H_1,H_2)$ into $\pi_2(\gamma)X\pi_1(\gamma^{-1})$.*

For simplicity we assume that Γ has infinite conjugacy non-trivial classes, so that its associated von Neumann algebra has a unique trace τ.

Then the $\pi(\Gamma)$ - invariant vectors in $\mathcal{C}_2(H_1, H_2)$ are identified with the L^2 - space of Γ - intertwiners of the representations π_1, π_2. We denote the space of Γ-intertwiners by $\mathrm{Int}_\Gamma(\pi_1, \pi_2)$ (If $H_1 = H_2, \pi_1 = \pi_2$ then this space is simply the commutant algebra $\pi_1(\Gamma)' \subseteq B(H_1)$, where and the L^2 - space is canonically determined by the trace on the von Neumann factor $\pi_1(\Gamma)'$).

In general the L^2 - space of $\mathrm{Int}_\Gamma(\pi_1, \pi_2)$, denoted bellow by $L^2(\mathrm{Int}_\Gamma(\pi_1, \pi_2), \tau)$, is obtained by first fixing one of the two spaces, which has greater equal Murray von Neumann dimension. Assume, for example,that this space is H_1. Then the L^2 - norm of X is $\tau_{\pi_1(\Gamma)'}(X^*X)^{1/2}$.

The classical Hecke operator on Γ invariant vectors associated to the representation π is then the extension to the L^2 space of intertwiners, of the operator $\Psi(\Gamma\sigma\Gamma)$ defined by the following formula, for $\sigma \in G$, (assuming the disjoint decomposition of the coset $\Gamma\sigma\Gamma$ is $\cup_i s_i\sigma\Gamma$):

$$\Psi(\Gamma\sigma\Gamma)(X) = \sum_i \pi_2(s_i\sigma)X\pi_1(\sigma^{-1}s_i^{-1}), X \in \mathrm{Int}_\Gamma(\pi_1, \pi_2).$$

This is obviously a bounded operator.

We identify $\mathrm{Int}_\Gamma(\pi_1, \pi_2)$ with $R(\Gamma) \otimes B(L_1, L_2)$ and we identify the associated Hilbert space $L^2(\mathrm{Int}_\Gamma(\pi_1, \pi_2), \tau)$ with $l^2(\Gamma) \otimes B(L_1, L_2)$. Let $S_i : \mathcal{H}_0 \to R(G) \otimes B(L_i)$, for $i = 1, 2$, be the representation of the Hecke algebra associated with the representations π_1, π_2 constructed in Proposition 2. Let S be the representation of the Hecke algebra $\mathcal{H}_0$, given by Proposition 2, associated to the representation π.

Then, for every double coset $\Gamma\sigma\Gamma$, the operator $\widetilde{\varepsilon}(S(\Gamma\sigma\Gamma))$ is $\Psi(\Gamma\sigma\Gamma)$ In particular, the construction in Proposition 2 for the representation π, yields the classical Hecke operators on $\mathrm{Int}_\Gamma(H_1, H_2)$. Moreover we have the following formula for $\Psi(\Gamma\sigma\Gamma)$:

$$\Psi(\Gamma\sigma\Gamma)(X) = E^{R(G)\otimes B(L_1,L_2)}_{R(\Gamma)\otimes B(L_1,L_2)}(S_2(\Gamma\sigma\Gamma)XS_1(\Gamma\sigma\Gamma)),$$

for all

$$X \in \mathrm{Int}_\Gamma(\pi_1, \pi_2) \cong R(\Gamma) \otimes B(L_1, L_2).$$

Here $E^{R(G)\otimes B(L_1,L_2)}_{R(\Gamma)\otimes B(L_1,L_2)}$ is the canonical conditional expectation.

Proof. We identify H_i with $l^2(\Gamma) \otimes L_i$. Then we have that $H_1 \otimes \overline{H}_2 = l^2(\Gamma) \otimes \overline{l^2(\Gamma)} \otimes L_1 \otimes \overline{L}_2$ and hence, a possible Γ wandering subspace for $H_1 \otimes \overline{H}_2$ is $l^2(\Gamma) \otimes L_1 \otimes \overline{L}_2$. This space is identified with $L^2(\mathrm{Int}_\Gamma(H_1, H_2), \tau)$ as follows:

Fix l_i, m_i vectors in $L_i, i = 1, 2$. Fix γ_a, γ_b in Γ. Let e be the identy element of the group Γ. Consider the following 1-dimensional operators: $A_0 =$

$\langle e\otimes l_1,\cdot\rangle(\gamma_a\otimes l_2)$, $B_0=\langle e\otimes m_1,\cdot\rangle(\gamma_b\otimes m_2)$. Let $A=\sum\pi_1(\gamma)A_0\pi_2(\gamma)^{-1}$, $B=\sum\pi_1(\gamma)B_0\pi_2(\gamma)^{-1}$. Then $\tau(AB^*)=\mathrm{Tr}(AB_0)$.

We obtain that $\Psi(\Gamma\sigma\Gamma)(A)=\sum\limits_{\theta\in\Gamma\sigma\Gamma}\pi_2(\theta)A_0\pi_1(\theta)^{-1}$ and the matrix coefficients are

$$\sum_{\theta\in\Gamma\sigma\Gamma}\mathrm{Tr}(\pi_2(\theta)\left[\langle e\otimes l_1,\cdot\rangle(\gamma_a\otimes l_2)\right]\pi_1(\theta)^{-1}(\langle e\otimes m_1,\cdot\rangle\gamma_b\otimes m_2)^*)=$$

$$=\sum_{\theta\in\Gamma\sigma\Gamma}\mathrm{Tr}(\left[\langle\pi_1(\theta)^{-1}(e\otimes l_1),\cdot\rangle\pi_2(\theta)(\gamma_a\otimes l_2)\right]\left[\langle e\otimes m_1,\cdot\rangle(\gamma_b\otimes m_2)\right]^*)=$$

$$=\sum_{\theta\in\Gamma\sigma\Gamma}\langle\pi_1(\theta)^{-1}(e\otimes l_1),\gamma_a\otimes l_2\rangle\langle\pi_2(\theta)(e\otimes m_1)(\gamma_b\otimes m_2)\rangle.$$

But this are exactly the matrix coefficients of $\widetilde{\varepsilon}(S)(\Gamma\sigma\Gamma)$ where S is the representation of $\mathcal{H}_0$ associated to the representation $\pi_1\otimes\overline{\pi}_2$ on the vectors in the Γ - wandering subspace $l^2(\Gamma)\otimes L_1\otimes\overline{L}_2\cong e\otimes\overline{l_2(\Gamma)}\otimes L_1\otimes\overline{L}_2$, corresponding to the vectors $e\otimes\gamma_a\otimes l_1\otimes l_2$ and $e\otimes\gamma_b\otimes m_1\otimes\overline{m}_2$.

To verify the formula for the conditional expectation, note that it is sufficient to check that we get the same values for the matrix entries, when evaluated at elements A,B as above.

The trace on $R(\Gamma)\otimes B(L_1)$ is $\tau_{R(\Gamma)}\otimes\mathrm{tr}_{B(L_1)}$.

Thus we have to compute for $A,B\in R(\Gamma)\otimes B(L_1,L_2)$,

$$\tau_{R(\Gamma)\otimes B(L_1)}(E^{R(G)\otimes B(L_1,L)}_{R(\Gamma)\otimes B(L_1,l_2)}(S_2(\Gamma\sigma\Gamma)AS_1(\Gamma\sigma\Gamma)B^*))=$$
$$=\tau_{R(\Gamma)\otimes B(L_1)}(S_2(\Gamma\sigma\Gamma)AS_1(\Gamma\sigma\Gamma)B^*)=$$
$$=\tau((\sum_{\theta_1\in\Gamma\sigma\Gamma}\rho(\theta_1^{-1})\otimes P_L\pi_2(\theta_1)P_L)(R_{\gamma_a}\otimes\langle l_1,\cdot\rangle l_2)$$
$$(\sum_{\theta_2\in\Gamma\sigma\Gamma}\rho(\theta_2^{-1})\otimes P_L\pi_1(\theta_2)P_L)R_{\gamma_b^{-1}}\otimes\langle m_1,\cdot\rangle m_2)=$$
$$=\sum_{\substack{\theta_1,\theta_2\in\Gamma\sigma\Gamma\\ \theta_1\gamma_a\theta_2\gamma_b^{-1}=e}}\mathrm{Tr}((P_L\pi_2(\theta_1)P_L)(\langle l_1,\cdot\rangle l_2)P_L\pi_1(\theta_2)P_L\langle m_1,\cdot\rangle m_2)=$$

(as $\theta_2=\gamma_b\theta_1^{-1}\gamma_a$)

$$=\sum_{\theta\in\Gamma\sigma\Gamma}\langle P_L\pi_2(\theta)P_Ll_2,m_2\rangle\overline{\langle P_L\pi_1(\gamma_a\theta_1^{-1}\gamma_b)P_Ll_1,m_1\rangle}=$$
$$=\sum_{\theta\in\Gamma\sigma\Gamma}\langle\pi_2(\theta)l_2,m_2\rangle\overline{\langle\pi_1(\gamma_a^{-1}\theta_1\gamma_b^{-1})l_1,m_1\rangle}$$

which is exactly the previous formula. $\square$

2. CHAPTER 2

In this section we extend the results in the previous section to the case of non-integer Murray von Neumann dimension. We asume that there exists a representation (π_0, Γ, G, H_0) as in the previous section, but we do not assume that there exists a splitting $H_0 = l^2(\Gamma) \otimes L_0$.

Instead we assume that there exists a representation (π, G) on a larger Hilbert space H, which has a splitting $H = l^2(\Gamma) \otimes L$, with $\pi|_\Gamma = \lambda_\Gamma \otimes \mathrm{Id}_L$, and such that if $p : H \to H_0$ is the canonical orthogonal projection, then $\pi(G)$ commutes with p and $\pi_0(g) = \pi(g)|_{H_0}$. Also we assume pP_L is trace class. Denote by $S : \mathcal{H}_0 \to R(\widehat{G) \otimes B}(L)$ the representation of $\mathcal{H}_0$ constructed in the previous chapter.

We will prove that p also commutes with S , and that $pS(\Gamma\sigma\Gamma)p$ is a representation of $\mathcal{H}_0$ into $p(R(\Gamma) \otimes B(L))p \cong \pi_0(\Gamma)'$ with the required properties (The only non-similar property is the fact that that the representation of $\mathcal{K}^*$ take values into $p(R/\Gamma) \otimes B(L))$.

Lemma 7. *Let π, G, H, L as above, and let the subspace $H_0 = pH$, where p is a projection such that $[p, \pi(g)] = 0$, for all g in G. Let $\pi_0(g) = \pi(g)|_{H_0}, g \in G$. We assume also that pP_L is trace class P_L.*

Then p commutes with $S(\Gamma\sigma\Gamma)$, for σ in G. The explicit expression for p is

$$\sum \rho(\gamma) \otimes P_L \pi_0(\gamma^{-1}) P_L = E_{(\pi_0(\Gamma))'}(pP_L) = E_{\pi_0(\Gamma)'}(pP_L p).$$

In this case, since pP_L is trace class, the above expression belongs to $\mathbb{R}(\Gamma) \otimes B(L)$.

Then

$$pS(\Gamma\sigma) = \sum_{\theta \in \Gamma\sigma} \rho(\theta^{-1}) \otimes P_L \pi_0(\theta) P_L \in p(R(G) \otimes B(L)),$$

and

$$pS(\Gamma\sigma\Gamma) = S(\Gamma\sigma\Gamma)p = \sum_{\theta \in \Gamma\sigma\Gamma} \rho(\theta^{-1}) \otimes P_L \pi_0(\theta) P_L$$

and hence $S_p(\Gamma\sigma\Gamma) = pS(\Gamma\sigma\Gamma)p = pS(\Gamma\sigma\Gamma)$ determines a representation of $\mathcal{H}_0$ while $\Gamma\sigma \to pS(\Gamma\sigma)$ determines a representation o $\mathcal{K}^*\mathcal{K}$ with $S(\sigma_1\Gamma)pS(\Gamma\sigma_2) = \sum\limits_{\theta \in \sigma_1\Gamma\sigma_2} \rho(\theta^{-1}) \otimes P_L \pi_0(\theta) P_L$.

Proof. The expression for p follows from the statement of Proposition 2 in chapter 1.

Then all the statements are a consequence of the formula

$$\sum_{\gamma} P_L \pi_0(\gamma) P_L \pi(\gamma^{-1}\theta) P_L = P_L \pi_0(\theta) P_L.$$

Indeed, for example, when doing

$$pS(\Gamma\sigma) = (\sum_{\gamma\in\Gamma} \rho(\gamma^{-1}) \otimes P_L \pi_0(\gamma) P_L)(\sum_{\theta\in\Gamma\sigma} \rho(\theta^{-1}) \otimes P_L \pi(\theta) P_L) =$$

$$= \sum_{\theta_1\in\Gamma\sigma} \rho(\theta_1^{-1}) \otimes \sum_{\substack{\theta\in\Gamma\sigma \\ \gamma\theta=\theta_1}} P_L \pi_0(\gamma) P_L \pi(\theta) P_L =$$

$$= \sum_{\theta_1\in\Gamma\sigma} \rho(\theta_1^{-1}) \otimes \sum_{\gamma\in\Gamma} P_L \pi_0(\gamma) P_L \pi(\gamma^{-1}\theta_1) P_L =$$

$$= \sum_{\theta_1\in\Gamma\sigma} \rho(\theta_1^{-1}) \otimes P_L \pi_0(\theta_1) P_L.$$

To check that $S_p(\Gamma\sigma\Gamma) = pS(\Gamma\sigma\Gamma)p$ is indeed a representation of $\mathcal{H}$ we have to verify that it implements the Hecke operators $\Psi_{\Gamma\sigma\Gamma}$ on $\pi_0(\Gamma)'$.

Here $\Psi_{[\Gamma\sigma\Gamma]}$ is described as the transformation mapping $X' \to \sum \pi(s_i\sigma)X'\pi(\sigma s_i)^{-1}$, for X' in $\pi_0(\Gamma)'$ (which as we mention before, extending by continuity to the L^2 - space is the application of $\widetilde{\varepsilon}$ to the Hecke operators for $H \otimes H$)and $\Gamma\sigma\Gamma = \cup s_i\sigma\Gamma$ a double coset.

Here $\pi_0(\Gamma)'$ is isomorphic (as p belongs to $R(\Gamma) \otimes B(L)$) to $p\pi_0(\Gamma)'p = p(R(\Gamma) \otimes B(L))p$. $\square$

We have

Theorem 8. *Let π_0, π, p be as above. Let $S_p : \mathcal{H}_0 \to p(R(G) \otimes B(L))p$ be the representation constructed in the previous proposition. Then we have*

$$E^{p(R(G)\otimes B(L))p}_{p(R(\Gamma)\otimes B(L))p}(S_p(\Gamma\sigma\Gamma)XS_p(\Gamma\sigma\Gamma)) =$$

$$= \Psi_{[\Gamma\sigma\Gamma]}(X) \; for \; all \; X \; in \; p(R(\Gamma) \otimes B(L))p.$$

Proof. Let $\sigma \in G$ and assume that $\Gamma\sigma\Gamma = \cup\Gamma\sigma s_i$, as a disjoin union.

Then $\Psi_{[\Gamma\sigma\Gamma]}(X) = \sum_i \pi_0(\Gamma s_i)^{-1} X \pi_0(\sigma s_i)$.

An element X' in $\pi_0(\Gamma)'$ has the following representation in $R(G) \otimes B(L)$

$$\sum \rho(\gamma^{-1}) \otimes P_L \pi_0(\gamma) X' P_L.$$

Thus we want to compute first the product

$$\sum_{\substack{\theta_1,\theta_2\in\Gamma\sigma\Gamma\\ \gamma\in\Gamma}}(\rho(\theta_1^{-1})\otimes P_L\pi_0(\theta_1)P_L)(\rho(\gamma^{-1})\otimes P_L\pi_0(\gamma)X'P_L)(\rho(\theta_2^{-1})P_L\pi_0(\theta)_2)P_L).$$

But this is equal to

$$\sum_{\theta\in(\Gamma\sigma\Gamma)^2}\rho(\theta^{-1})\otimes\sum_{\substack{\theta_1,\theta_2\in\Gamma\sigma\Gamma,\ \gamma\in\Gamma\\ \theta_1\gamma\theta_2=\theta}}P_L\pi_0(\theta_1)P_L\pi_0(\gamma)X'P_L\pi_0(\theta_2)P_L.$$

In the second sum we necessary have $\theta_1=\theta\theta_2^{-1}\gamma^{-1}$. Hence the above sum becomes

$$\sum_{\theta\in(\Gamma\sigma\Gamma)^2}\rho(\theta^{-1})\otimes\sum_{\theta_2\in\Gamma\sigma\Gamma}P_L\pi_0(\theta\theta_2^{-1})X'P_L\pi_0(\theta_2)P_L. \qquad (*)$$

We use the decomposition $\Gamma\sigma\Gamma=\cup\Gamma\sigma s_i$.

Then the sum of terms for a fixed σs_i corresponds to

$$\sum_{\gamma\in\Gamma}P_L\pi_0(\theta)\pi_0(\sigma s_i)^{-1}\pi_0(\gamma)X'P_L\pi_0(\gamma)\pi_0(\sigma s_i)$$

which since $[X',\pi_0(\gamma)]$, and again since $\sum\limits_{\gamma}\pi_0(\gamma)P_L\pi_0(\gamma)^{-1}=p$ gives

$$P_L\pi_0(\theta)\pi_0(\sigma S_i)^{-1}X'\pi_0(\sigma S_i)P_L=P_L\pi_0(\theta)\Psi_{[\Gamma\sigma\Gamma]}(X')P_L.$$

Thus

$$S_p(\Gamma\sigma\Gamma)X'S_p(\Gamma\sigma\Gamma)=\sum_{\theta\in(\Gamma\sigma\Gamma)^2}\rho(\theta^{-1})\otimes P_L\pi_0(\theta)\Psi(X')P_L.$$

Note that by $(\Gamma\sigma\Gamma)^2$ we simply understand the set $\Gamma\sigma\Gamma\sigma\Gamma$ with no multiplicities (as in the sum after θ, every θ appears only once).

Now an easy argument of the same type shows that for $X',Y'\in\pi_0(\Gamma)'$

$$A\cdot B=(\sum_{\theta\in\sigma\Gamma}\rho(\theta^{-1})\otimes P_L\pi_0(\theta)Y'P_L)(\sum_{\gamma\in\Gamma}\rho(\gamma^{-1})\otimes P_L\pi_0(\gamma)X'P_L)=$$

$$=\sum_{\theta_1\in\sigma\Gamma}\rho(\theta_1^{-1})\otimes P_L\pi_0(\theta_1)Y'X'P_L$$

thus the product AB has zero trace, and thus the two sums corresponding to the two factors in the product AB above are orthogonal.

Hence

$$E^{p(R(G)\otimes B(L))p}_{p(R(\Gamma)\otimes B(L))p}(\sum_{\theta\in(\Gamma\sigma\Gamma)^2}\rho(\theta^{-1})\otimes P_L\pi_0(\theta)\Psi(X')P_L)=$$

$$= \sum_{\gamma \in \Gamma} \rho(\gamma^{-1}) \otimes P_L \pi_0(\gamma) \Psi(X') P_L$$

which is exactly $\Psi(X')$. □

We observe that the representation of $\mathcal{H}_0$ obtained by this method. in the case $\dim_{\pi_0(\Gamma)} H_0$ an integer, is the same as the one obtained in the previous chapter.

We prove the following lemma of abstract character. It will be used to prove that the representation S_p is the same as the representation we constructed in ([Ra]). This technique can also be used to evaluate traces of Hecke operators as we explain bellow.

Lemma 9. *Let M be a type II_1 factor with trace τ and let $\widetilde{\mathcal{M}} = M \otimes B(H)$ be the associated type II_∞ factor with trace $T = \tau \otimes \mathrm{Tr}$. Let N be the subfactor $M \otimes I$ a subfactor and let p be a projection in $\widetilde{\mathcal{M}}$.*

Assume that $E_N(p)$ is invertible (which as we will observe automatically implies $T(p) = 1$).

Then the map

$$\Phi : p\widetilde{\mathcal{M}}p \to M$$

defined by

$$\Phi(pmp) = E_N(p)^{-1/2} E_N(pmp) E_N(p)^{-1/2},\ pmp \in pMp$$

is an algebra isomorphism.

Proof. We consider the Jones basic construction algebra $\widetilde{\mathcal{M}}_1 = \langle \widetilde{\mathcal{M}}, e_N \rangle$ assocaited to $N \subseteq M \otimes 1$, with trace $\widetilde{T}$.

Then e_N commutes with N and the map $x \in N \to e_N x$ is as isomorphism.

Then, by using thus isomorphism, the statement is equivalent to prove that the map Φe_N on pMp associating:

$$pmp \to (e_N p e_N)^{-1/2} e_N p (pmp) e_N p (e_N p e_N)^{-1/2}$$

is an isomorphism.

But $v = (e_N p e_N)_N^{-1/2} p$ is the partial isometry from the polar decomposition of $e_N p$, and hence this $e_N p e_N$ is invertible it is an isometry from p onto e_N (Since $\widetilde{T}(e_N) = 1$ necessary $\widetilde{T}(p) = T(p) = 1$).

Hence $\Phi e_N = \mathrm{Ad} v : p\widetilde{M}p \to e_N \widetilde{M} e_N = N e_N$ and hence Φ is an algebra isomorphism. □

Observation 10. If $E_N(p)$ is not an invertible element we assume that there exist positive scalars λ_i, with $\sum \lambda_i = 1$, and unitaries $u_i \in M$ identified with $u_i \otimes 1$ such that $\sum \lambda_i u_i E_N(p) u_i^*$ is invertible. We then the replace $\widetilde{M}$ by $\overset{\approx}{M} = \overset{i=1}{\underset{n}{\oplus}} \widetilde{M}$ where each component has weight $\lambda_i^{1/2}$ and considering the embedding $M \otimes 1 \subseteq \overset{\approx}{M}$, through the maps $m \otimes 1 \to \oplus\, u_i m u_i^* \otimes 1$.

Then the conditional expectation of $\widetilde{p} = \overset{i=1}{\underset{n}{\oplus}} p \in \overset{\approx}{M}$ is $\sum \lambda_i u_i^* E_N(p) u_i$ and hence is invertible. Hence we can still apply the previous lemma to this representation.

We consider first the one dimensional (in the sense of the Murray and von Neumann dimension theory). In this case $\dim_{\pi(\Gamma)} H_0 = 1$. If η is a cyclic trace vector, then we can choose a subspace $L_0 = \mathbb{C}\eta$ and hence the projection P_{L_0} would be $\langle \eta, \cdot \rangle \eta$ and hence the map $S : \mathcal{H} \to R(G) \otimes \mathbb{C}\langle \eta, \cdot \rangle \eta$ is

$$S(\Gamma\sigma\Gamma) = \sum_{\theta \in \Gamma\sigma\Gamma} \rho(\theta) \otimes P_{L_0} \pi_0(\theta) P_{L_0} =$$

$$= \sum_{\theta \in \Gamma\sigma\Gamma} \rho(\theta) \otimes \langle \pi(\theta)\eta, \eta \rangle P_{L_0}.$$

Identifying $\mathbb{C}P_{L_0}$ with $\mathbb{C}$, we get exacly the representation in [Ra].

Proposition 11. *With the notation from the pevious proposition, since* $p = \sum \rho(\gamma^{-1}) \otimes P_{L_0} \pi_0(\gamma) P_L$ *we have*

$$E_{R(G)\otimes 1}(p) = \sum \rho(\gamma^{-1}) \mathrm{Tr}(P_L \pi_0(\gamma))$$

$$E_{R(G)\otimes 1}(S_p(\Gamma\sigma\Gamma)) = \sum_{\theta \in \Gamma\sigma\Gamma} \rho(\theta) \mathrm{Tr}(P_L \pi_0(\theta)).$$

Let $N = R(\Gamma) \otimes 1$ *and let* $\zeta = E_N(p)^{-1/2}$. *If* $E_N(p)$ *is invertible then the map on* $\mathcal{H}$, *mapping* $[\Gamma\sigma\Gamma]$ *into* $S_p(\Gamma\sigma\Gamma)$, *constructed in this chapter is unitarely equivalent, by Lemma 9, to the representation*

$$[\Gamma\sigma\Gamma] \to \zeta^{-1/2} \sum_{\theta \in \Gamma\sigma\Gamma} \rho(\theta) \mathrm{Tr}(P_L \pi_0(\theta)) \zeta^{-1/2} \qquad (*)$$

where for $\sigma \in G$, *we make the hypothesis that* ζ *is invertible.*

If $E_{R(\Gamma)\otimes 1}(p)$ *is not invertible, by [BH] we may find* $g_1, \dots, g_n \in \Gamma$ *and positive scalars* λ_i, $\sum \lambda_i = 1$ *such that*

$$\sum_i \lambda_i (\rho(g_i) \otimes 1) E_{R(\Gamma)\otimes 1}(p) (\rho(g_i)^{-1} \otimes 1)$$

is invertible, and we can use the Observation 10, to obtain a representation of the algebra $\mathcal{H}$.

Proof. This follows from the previous two results.

Remark 12. The representation $*$ is exactly the map corresponding to $[\Gamma\sigma\Gamma] \to \sum_{\theta\in\Gamma\sigma\Gamma} \rho(\theta)\langle\pi(\theta)\eta,\eta\rangle$ where η is a cyclic trace vector $P_L p P_L = \sum\langle\zeta_i,\cdot\rangle\zeta_i$, where ζ_i is an orthogonal basis. Let $\widetilde{\zeta_i} = \lambda_i^{1/2}\zeta_i$, which is an orthogonal system. Then $\widetilde{\eta} = \oplus\ \widetilde{\zeta_i} \in H_0^{\aleph_0} = \widetilde{H}_0$ is an unit vector. Let $\widetilde{\pi} = \oplus\ \pi$ acting diagonally on $\widetilde{H}_0$.

Then $\langle\widetilde{\pi}(\theta)\widetilde{\eta},\widetilde{\eta}\rangle = \mathrm{Tr}(P_L\pi_0(\theta)) = \mathrm{Tr}(pP_Lp\pi(\theta))$.

Proof. Then $\widetilde{\eta}$ is not necessary a Γ - wandering vector, but $\widetilde{\eta}_0 = \pi(\zeta)^{-1/2}\widetilde{\eta}$ is a Γ - wandering vector and, because $\widetilde{\pi}$ is a multiple of π, it follows that $Sp\overline{\widetilde{\pi}(\gamma)\widetilde{\eta}}$ is invariant under $\widetilde{\pi}(g)$, $g \in G$. Thus the coefficients of the representation of $\mathcal{H}_0 \to H$ as in Chapter 1, associated to π, are

$$[\Gamma\sigma\Gamma] \to \sum \rho_\theta\langle\widetilde{\pi}(\widetilde{\eta}_0),\widetilde{\eta}_0\rangle \in R(G).$$

But this is exactly $\zeta^{-1/2}\sum\rho_\theta\mathrm{Tr}(P_L\pi_0(\theta))\zeta^{-1/2} \in R(G)$, for $\sigma \in G$. $\square$

In the case $n > 1$, $\dim_{\pi(\Gamma)}H_0 = n$, we consider only the case $H = L^2(X,\mu)$, $L = L^2(F,\mu)$.

Here (X,μ) is an infinite measure space. Assume that G acts on X by measure preserving transformations and F is a fundamental domain for Γ in X.

We divide F into n equal parts. We assume that v_{i1}, the isometry from the polar decomposition of $\chi_{F_i}p\chi_{F_1}$ is invertible.

Then $\pi(\Gamma)'$ is isomorphic to $R(\Gamma)\otimes M_n(\mathbb{C})\otimes L^2(F_1)$ and a given element X' in $\pi(\Gamma)'$ is represented as

$$\sum_{\gamma\in\Gamma}\rho(\gamma^{-1})\otimes(\chi_{F_1}v_{1i}\pi_0(\gamma)X'v_{j1}\chi_{F_1})_{ij}$$

while

$$S_p(\Gamma\sigma\Gamma) = \sum_{\theta\in\Gamma\sigma\Gamma}\rho(\theta^{-1})\otimes(\chi_{F_1}v_{1i}\pi_0(\theta)v_{j1}\chi_{F_1})_{ij},\ \sigma\in G.$$

Applying the previous lemma, and assuming that $E_{R(\Gamma)\otimes M_n(\mathbb{C})}(p)$ is invertible, we get that thus representation of $\mathcal{H}$ is unitary equivalent to the representation S_0 mapping $[\Gamma\sigma\Gamma]$ into $\sum\limits_{\theta\in\Gamma\sigma\Gamma}\rho(\theta)\otimes \mathrm{Tr}(\chi_F v_{1i}\pi_0(\theta)v_{j1}\chi_{F_1})$, for $\sigma\in G$.

Hence $\varepsilon(S_0(\Gamma\sigma\Gamma))$ is $\sum\limits_{\theta\in\Gamma\sigma\Gamma}(\mathrm{Tr}(v_{j1}v_{1i}\pi_0(\theta)))_{ij}$.

Thus the trace of the Hecke operators is

$$\mathrm{Tr}(\mathrm{Tr}((v_{j1}v_{1j})\sum\pi_0(\theta))_{ij}).$$

This is similar to the formula in [Za]. We remark that for $n=13$, $\pi=\pi_{13}$, the 13-th element in the discrete series of $PSL_2(\mathbb{R})$ on H_{13}, $G=PGL_2(\mathbb{R}[\frac{1}{p}])$, $\Gamma=PSL_2(\mathbb{Z})$, the trace of the Hecke operator on "Γ - invariant" vectors in H_{13} results to be

$$\sum_{\theta\in\Gamma\sigma\Gamma}\mathrm{Tr}(\pi_0(\theta)\chi_F)=\sum_{\theta\in\Gamma\sigma\Gamma}\int_F(\frac{1}{1-\bar{z}\theta(z)})^t d\Gamma_t(z).$$

(see also next chapter for an interpretation).

3. Chapter 3

In this section we determine the relation between the action of the Hecke operators on Γ invariant vectors and the representation of the Hecke operators constructed in the previous chapter. This is done in the absence of a splitting space for the given representation (i.e. of the type $H_0=l^2(\Gamma)\otimes L_0$), but in the presence of a splitting for a larger representation of G, that contains the given representation as a superrepresentation. We prove that the representations and traces of Hecke operators are obtained by applying the (unbounded) character $\varepsilon: R(G)\to\mathbb{C}$ (the sum of coefficients) to the representations of the Hecke algebra constructed in Chapter 1 and 2.

Let $H_0\subseteq H$ be Hilbert space, $\pi: G\to\mathcal{U}(H)$ a rpresentation invariating H_0 and let $\pi_0=\pi|_{H_0}$.

Denote by p the projection onto H_0. We assume that L is a Γ - wandering, generating subspace of H, thus $H\cong l^2(\Gamma)\otimes L$, with L identified with $e\otimes L$, and $\pi=\lambda_\Gamma\otimes\mathrm{Id}_L$.

We use L as a scale to measure the Γ - invariant ("unbounded") vectors in H_0 (this is exactly the Petersson scalar product).

Definition 13. Let H_0^Γ be the space of densely defined, Γ - invariant (including Γ - invariant domain) functionals on H_0^Γ. Let $H_0^\Gamma(L,\pi)$ be the linear subspace of $v\in H_0^\Gamma$, such that the vector $P_L v$ (defined by $\langle P_L v, w\rangle =$

$\langle v, P_L w\rangle_H$ for $w \in \mathrm{Dom}(v)$), is a bounded linear form on H_0. $H_0^\Gamma(L,\pi)$ will consist of v such that $\langle P_L v, v\rangle_H < \infty$.

Thus $H_0^\Gamma(\pi, L)$ is a Hilbert space with scalar product $\langle v_1, v_2\rangle_{H_0^\Gamma(\pi,L)} = \langle P_L v_1, v_2\rangle_H$.

Observation 14. If the vector v_2 in H_0^Γ has the property that there exists $w_2 \in H_0$ such that $v_2 = \sum\limits_{\gamma\in\Gamma} \pi_0(\gamma) w_2$ (in the sense of pointwise convergence on a dense subset, which is fulfilled for example if w_2 is a Γ - wandering vector), then

$$\langle v_1, v_2\rangle_{H_0^\Gamma(\pi,L)} = \langle P_L v_1, \sum_{\gamma_2} \pi(\gamma_2) w_2\rangle_H =$$

$$= \langle \sum \pi(\gamma_2) P_L v_1, v_2\rangle_H = \sum \langle \pi(\gamma_2) p \pi(\gamma_2^{-1}) v_1, w_2\rangle_H = \langle v_1, w_2\rangle_H.$$

We consider the following linear map from L onto $H_0^\Gamma(\pi, L)$

$$\Phi(l) = \sum_\gamma \pi_0(\gamma) l = p(\sum \pi(\gamma) l).$$

It is well defined as $\sum \pi(\gamma) l$ defines densely defined Γ - invariant functionals on H.

Proposition 15. *Assume $H_0^\Gamma(\pi, L)$ is finite dimesional and $P_L p$ is the trace class. Let $A(\Gamma\sigma\Gamma) = \sum\limits_{\theta\in\Gamma\sigma\Gamma} P_L \pi_0(\theta) P_L$, $\sigma \in G$.*

Then $A(\Gamma\sigma\Gamma) = \widetilde{\varepsilon}(S_p(\Gamma\sigma\Gamma))$, $\sigma \in G$, where $\widetilde{\varepsilon}$ is the restriction to $p(R(G) \otimes B(L))p$ of the unbounded character $\varepsilon \otimes \mathrm{Id}$, which is the unbounded character on $R(G) \otimes B(L)$ equal to the sum after G of the operational coefficients. Here S_p is the representation constructed in Chapter 2. In particular $A(\Gamma\sigma\Gamma)$ is a representation of the Hecke algebra, $A(\Gamma) = \widetilde{\varepsilon}(p)$ is a projection, and $A(\Gamma\sigma\Gamma) = \widetilde{\varepsilon}(p) A(\Gamma\sigma\Gamma) \widetilde{\varepsilon}(p)$.

Moreover $A(\Gamma) = \Phi^*\Phi$, and hence $A(\Gamma)$ is a finite dimensional projection .

Moreover if $\widetilde{H}_0^\Gamma(L,\pi)$ is the image of Φ in $H_0^\Gamma(L,\pi)$, then Φ intertwines the action of the Hecke operators $T_{H_0}(\Gamma\sigma\Gamma)$ on $\widetilde{H}_0^\Gamma(L,\pi)$ with te representation $[\Gamma\sigma\Gamma] \to A(\Gamma\sigma\Gamma)$ into $B(\widetilde{\varepsilon}(p)L)$.

In particular

$$\mathrm{Tr}(T_{H_0}(\Gamma\sigma\Gamma)|_{\widetilde{H}_0^\Gamma(L,\pi)}) = \sum_{\theta\in\Gamma\sigma\Gamma} \mathrm{Tr}(P_L \pi_0(\theta) P_L)$$

(which is the formula used in Zagie's paper ([Za]) on the Eichler Selberg trace formula).

Proof. Clearly for $l_1, l_2 \in L$

$$\langle \Phi(l_1), \Phi(l_2) \rangle = \langle \sum_{\gamma_1} \pi_0(\gamma_1) l_1, \sum_{\gamma_2} \pi_0(\gamma) l_2 \rangle_{H_0^\Gamma(Pi,L)}.$$

This, as we observed in the preceding remark, is equal to

$$\langle \sum_{\gamma} \pi_0(\gamma) l_1, l_2 \rangle = \sum_{\gamma} \langle P_L \pi_0(\gamma) P_L l_1, l_2 \rangle.$$

Thus $\Phi^*\Phi = \sum\limits_{\gamma} P_L \pi_0(\gamma) P_L$, but by our assumption the image of Φ is finite dimensional, thus the sum above is weakly convergent. The sane argument works for the similar sums defining $A(\Gamma\sigma\Gamma)$.

□

The intertwining formula for Φ is a consequence of the fact that the action of the Hecke operator $T(\Gamma\sigma\Gamma)$ on a vector of the form $v = \sum \pi_0(\gamma) w$ is $T(\Gamma\sigma\Gamma) v = \sum\limits_{\theta \in \Gamma\sigma\Gamma} \pi_0(\theta) w$.

Remark 16. Note that in the case of a representation π_0 coming from a unitary representation $\overline{\pi}_0$ on H_0 of a larger group $\overline{G}$ which contains G as a dense subgroup, and such that the representation (projective) π is in the discrete series of G, we have that

$$\mathrm{Tr}(P_L p) = \dim_{\pi_0(\Gamma)} H_0 = \frac{\text{card } F}{d_\pi}$$

(while as observed in the previous proposition

$$\dim_{\mathbb{C}} H_0^\Gamma(L, \pi) = \sum_{\gamma \in \Gamma} \mathrm{Tr}(P_L \pi_0(\gamma)) \text{).}$$

Proof. This is essentially the formula in [VFR]. For completness we reprove it here.

The Plancherel dimension formula gives that for A_0 a trace class operator in $\mathcal{C}(H_0)$, we have

$$\int_G \pi_0(g) A_0 \pi_0(g) = d_\pi \mathrm{tr}(A_0) p.$$

Because of the arguments in [VFR]

$$\dim_{\{\pi_0(\Gamma)\}''} H_0 = \mathrm{Tr}(P_L p) = \mathrm{Tr}(P_L p P_L) = \mathrm{Tr}(p P_L p).$$

But then

$$d_\pi p \mathrm{Tr}(p P_l p) = \int_G \pi_0(g) p P_l p \pi_0(g)^{-1} dg =$$

$$= \int_{G/\Gamma} \pi_0(g) \Big(\sum_{\gamma\ in \Gamma} \pi_0(\gamma) p P_L p \pi_0(\gamma)^{-1}\Big) \pi_0(g)^{-1} dg.$$

But $\sum\limits_{\gamma\in\Gamma} \pi_0(\gamma) p P_L p \pi_0(\gamma)^{-1} = p$ and hence the integral is further equal to

$$\int_{G/\Gamma} \pi_0(g) p \pi_0(g)^{-1} dg = p\ (\mathrm{covol}(\Gamma)).$$

$\square$

Remark 17. In the sum $\sum\limits_{\theta\in\Gamma\sigma\Gamma} \mathrm{Tr}(\pi_0(\theta) P_L)$, if we divide by the conjugacy equivalence orbits $\Gamma\sigma\Gamma/\sim$ of Γ in $\Gamma\sigma\Gamma$ we get

$$\sum_{\theta\in\Gamma\sigma\Gamma/\sim} \sum_{\gamma\in\Gamma} \mathrm{Tr}(\pi_0(\theta)\pi_0(\gamma) P_L \pi_0(\gamma)^{-1}).$$

The term, for $\theta \in G$,

$$\sum_{\gamma\in\Gamma} \mathrm{Tr}(\pi_0(\theta)\pi_0(\gamma) P_L \pi_0(\gamma^{-1}))$$

is a conjugacy invariant, equal to the coefficient, by which the conjugacy class of θ enters in the Plancherel formula for π. In fact, in the case of $\overline{G} = PGL_2(\mathbb{R})$, by using the Berezin ([Be]) symbol $\pi(\widehat{\theta})(\overline{z}, \eta)$ of the unitary $\pi(\theta)$, this sum is equal to

$$\sum_\gamma \int_{\gamma F} \widehat{\pi_0(\theta)}(\overline{z}, z) d\nu_0(z)$$

which is further equal to

$$\int_{\mathbb{H}} \widehat{\pi_0(\theta)}(\overline{z}, z) d\eta_0(z) = \int_{\mathbb{H}} \frac{1}{(1-\overline{z}\theta z)^t} (j(\theta, z))^t d\nu_t(\eta)$$

(which is the coefficient that shows up in Zagier formula [Za]).

4. CHAPTER 4

In this section we prove that our construction provides a realization of the Hecke operators as values of a matrix valued scalar product on the operators system $\mathcal{K}_0\mathcal{K}_0^*$ described in Chapter 1.

Recall that $\mathcal{K}_0^*\mathcal{K}_0 = \mathbb{C}(G/\Gamma) \underset{\mathcal{H}_0}{\otimes} \mathbb{C}(\Gamma \setminus G)$ and we define a scalar product on $\mathcal{K}_0\mathcal{K}_0^* = \mathbb{C}(G\setminus\Gamma)\otimes l^2(\Gamma\setminus G)$ with values in the algebra of Hecke operators.

The scalar product is compatible with the $\mathcal{H}_0$ module structure on $\mathcal{K}_0^*\mathcal{K}_0$.

Definition 18. Assume that π_0, G, H_0 is a representation of G as in the previous section. We make the additional ssumption that H_0 splits as $l^2(\Gamma) \otimes L_0$, where the Hilbert space L_0 is finite dimensional, and such that $\pi_0|_\Gamma$ is unitarely equivalent to $\lambda_\Gamma \otimes \mathrm{Id}_{L_0}$.

Let $S_0 : \mathcal{K}_0^*\mathcal{K}_0 \supseteq \mathcal{H}_0 \to R(G) \otimes B(L_0)$ be the representation of the Hecke algebra and the larger operator system $\mathcal{K}_0^*\mathcal{K}_0$ constructed in Chapter 1.

We will say that π_0 is tamed if the range of $S_0(\mathcal{K}_0^*\mathcal{K}_0)$ is contained in the domain of $\varepsilon \otimes \mathrm{Id}_{B(L_0)} : R(G) \otimes B(L_0) \to B(L_0)$ (In particular this means that the matrix coefficients of $S_0^{\sigma_1\Gamma} S_0^{\Gamma\sigma_2} = \sum\limits_{\theta\in\sigma_1\Gamma\sigma_2} \rho(\theta^{-1}) \otimes P_{L_0}\pi_0(\theta)P_{L_0}$ are in $L^1(\sigma_1\Gamma\sigma_2, B(L))$, for $\sigma_1, \sigma_2 \in G$).

Consider the densely defined linear operator $\mathcal{I}$ on $R(G)$ with values into $R_{G/\Gamma}(G)$ defined by $\mathcal{I}(\sum a_g\rho_g) = \sum a_g\rho_{G/\Gamma}(g)$, where $\rho_{G/\Gamma} : G \to B(l^2(G/\Gamma))$ is the right regular representation of G into $l^2(G/\Gamma)$.

Clearly our hypothesis implies that the domain of $\mathcal{I}$ contains the range of S.

Let $\widetilde{\varepsilon} : R_{G/\Gamma}(G) \to \mathbb{C}$ be the densely defined character given by the formula $\widetilde{\varepsilon}(\sum a_g\rho_{G/\Gamma}(g)) = \sum a_g \in \mathbb{C}$. Then the range of $\mathcal{I} \circ S$ is contained in the domain of $\widetilde{\varepsilon} \circ \mathrm{Id}_{B(L_0)}$, and we have the the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{K}_0^*\mathcal{K}_0 & \xrightarrow{S_0} & R(G)\otimes B(L_0) & \xrightarrow{\mathcal{I}\otimes \mathrm{Id}_{B(L_0)}} & R_{G/\Gamma}(G)\otimes B(L) \\
& \searrow^{T_0} & \downarrow \varepsilon\otimes \mathrm{Id}_{B(L_0)} & \swarrow \widetilde{\varepsilon} & \\
& & B(L_0) & &
\end{array}$$

Thus we obtain a representation $T_0 = (\varepsilon \circ \mathrm{Id}_{B(L_0)}) \circ S$ which extends the representation of Hecke operators from $\mathcal{H}_0$ to $\mathcal{K}_0^*\mathcal{K}_0$. The image of the representation T_0 consists of operators acting on the space of vectors in L_0. This latest space is canonically identified to Γ - invariant vectors in H_0.

The main reason for introducing the additional map $\mathcal{I}$ in the above diagram, is because, to compute values of T_0 on $\sigma_1\Gamma\sigma_2$ we do a sum over $\sigma_1\Gamma\sigma_2$ which corresponds to a matrix element in the factorization through $R_{G\setminus\Gamma}(G)$.

In fact

$$(\varepsilon \otimes \mathrm{Id}_{B(L_0)})(S(\sigma_1\Gamma\sigma_2)) = \sum_{\theta\in\sigma_1\Gamma\sigma_2} P_{L_0}\pi_0(\theta)P_{L_0}.$$

Indeed this follows from the fact that for $X = \sum a_g\rho_{G/\Gamma}(g)$, where a_g are complex coefficients, $L^1(G)$, we have

$$\langle\mathcal{I}(X)\sigma_1\Gamma, \sigma_2\Gamma\rangle = \sum_{g\in\sigma_2\Gamma\sigma_1^{-1}} a_g$$

and hence if $X = \sum\limits_{g\in\sigma_4\Gamma\sigma_3^{-1}} a_g\rho_{G/\Gamma}(g)$ then $\langle X\sigma_1\Gamma, \sigma_2\Gamma\rangle = \sum\limits_{g\in\sigma_4\Gamma\sigma_3^{-1}\cap\sigma_2\Gamma\sigma_1^{-1}} a_g$.

Proposition 19. *Thre exists a $B(L_0)$ valued, scalar product on $\mathcal{K}_0\mathcal{K}_0^* = l^2(G\setminus\Gamma)\otimes l^2(\Gamma\setminus G)$ with the following properties*

1) $\langle\Gamma\sigma_1\otimes\sigma_1\Gamma, \Gamma\sigma_2\otimes\sigma_2\Gamma\rangle = T_0(\sigma_1\Gamma)T(\Gamma\sigma_2) = T_0(\Gamma\sigma_1)^*T_0(\Gamma\sigma_2)$, $\sigma_1, \sigma_2 \in G$. In particular if we take a disjoint reunion of cosets of the form $\sigma_1\Gamma, \Gamma\sigma_2$ whose reunion is a double coset, we get the the Hecke operator corresponding to the double coset.

2) $\langle\Gamma\sigma_1\otimes\sigma_2\Gamma, \Gamma\sigma_3\otimes\sigma_4\Gamma\rangle$ depends only on $\sigma_3\Gamma\sigma_1\cap\sigma_4\Gamma\sigma_2$, for all $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in G$.

Equivalently by cyclically rotating the variables we obtain a bilinear form $\ll\cdot,\cdot\gg$ on $\mathcal{K}_0^*\mathcal{K}_0 = l^2(\Gamma\setminus G)\underset{\mathcal{H}}{\otimes} l^2(G/\Gamma)$, defined by the formula

$$\ll [\sigma_3\Gamma]\underset{\mathcal{H}_0}{\otimes}[\Gamma\sigma_1], [\sigma_4\Gamma]\underset{\mathcal{H}_0}{\otimes}[\Gamma\sigma_2] \gg = \langle\Gamma\sigma_1\otimes\sigma_2\Gamma, \Gamma\sigma_3\otimes\sigma_4\Gamma\rangle.$$

An equivalent form to describe this property of the scalar product is that for every $r \in \mathcal{H}_0$, we have

$$\langle\Gamma\sigma_1\otimes[(\sigma_2\Gamma)r], \Gamma\sigma_3\otimes\sigma_4\Gamma\rangle = \langle\Gamma\sigma_1\otimes\sigma_2\Gamma, \Gamma\sigma_3\otimes[(\sigma_4\Gamma)r]\rangle$$

for all $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in G$.

3) The support properties are preserved by the scalar product: For all $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in G$ if $\Gamma\sigma_1\sigma_2^{-1}\Gamma\cap\Gamma\sigma_3^{-1}\sigma_4\Gamma$ is valid, or $\sigma - \Gamma\sigma_1\cap\sigma_4\Gamma\sigma_2$ then the scalar product $\langle\Gamma\sigma_1\otimes\sigma_2\Gamma, \Gamma\sigma_3\otimes\sigma_4\Gamma\rangle = 0$.

Proof. With the previous notations, let $\widetilde{S}_0$ be the composition $\mathcal{I}\circ S_0$ which is a multiplicative map form $\mathcal{K}_0^*\mathcal{K}_0$ into $B(L_0)$.

By multiplicativity we understand

$$\widetilde{S}_0(k_1^*)\widetilde{S}_0(k_2) = \widetilde{S}_0(k_1)^*\widetilde{S}_0(k_2) = \widetilde{S}_0(k_1^*k_2)$$

for all $k_1, k_2 \in \mathcal{K}$.

Note that $\widetilde{S}_0(k)$ belongs to $R_{G/\Gamma}(G)$ which by ([Co], [Tz]) is the commutant in $B(l^2(G \setminus \Gamma))$ of the Hecke algebra $\mathcal{H}_r$, acting from the right on $l^2(G \setminus \Gamma)$.

For h, k vectors in $l^2(G/\Gamma)$ we let

$$\widetilde{w}_{h,k} : B(l^2(G/\Gamma) \otimes B(L_0)) \to B(L_0)$$

be the functional $\langle \cdot, h\rangle k \otimes \mathrm{Id}_{B(L_0)}$.

For simplicity for h, k in $l^2(G/\Gamma)$ we denote $\widetilde{w}_{h,k}(A) = \langle Ah, k\rangle$. Note that this is an element in $B(L_0)$.

Then $\langle \widetilde{S}(\Gamma\sigma_1)\sigma_2\Gamma, \widetilde{S}(\Gamma\sigma_3)\sigma_4\Gamma\rangle$ with the above notation is an element in $B(L_0)$.

Moreover because $\widetilde{S}$ is a representation of $\mathcal{K}_0^*\mathcal{K}_0$, this is further equal to

$$\langle \widetilde{S}(\sigma_3^{-1}\Gamma\sigma_1)\sigma_2\Gamma\sigma_4\Gamma\rangle.$$

By the previous observation this is

$$\sum_{\theta \in \sigma_3^{-1}\Gamma\sigma_1 \cap \sigma_4\Gamma\sigma_2^{-1}} P_{L_0}\pi_0(\theta)P_{L_0}.$$

We define then the scalar product by the formula

$$\langle \Gamma\sigma_1 \otimes \sigma_2\Gamma, \Gamma\sigma_3 \otimes \sigma_4\Gamma = \langle \widetilde{S}_0(\Gamma\sigma_1)[\sigma_2^{-1}\Gamma], \widetilde{S}_0(\Gamma\sigma_3^{-1})(\sigma_4\Gamma)\rangle.$$

This formula also proves the positive defitness of the scalar produc.

This is consequently equal to

$$\sum_{\theta \in \sigma_3\Gamma\sigma_1 \cap \sigma_4\Gamma\sigma_2} P_{L_0}\pi_0(\theta)P_{L_0}. \qquad (**)$$

This proves property (2). The equivalent form of property (2) follows from the fact that for all σ_3, σ_1 in G, $\widetilde{S}_0(\sigma_3^{-1}\Gamma\sigma_1)$ belongs to $\mathcal{H}_r \otimes B(L_0)$.

Property (1) now follows from the fact that

$$\langle \Gamma\sigma_1 \otimes \sigma_1\Gamma, \Gamma\sigma_2 \otimes \sigma_2\Gamma\rangle$$

is equal, by formula $(**)$ to

$$\sum_{\theta \in \sigma_1\Gamma\sigma_2} P_{L_0}\pi_0(\theta)P_{L_0}$$

which by the previous diagram is

$$T_0(\sigma_1\Gamma)T_0(\Gamma\sigma_2) = T_0(\Gamma\sigma_1)^*T_0(\Gamma\sigma_2)$$

for all $\sigma_1, \sigma_2 \in G$.

$\square$

Remark 20. The operators $T_0(\sigma_1\Gamma)T_0(\Gamma\sigma_2)$ are not classical Hecke operators, but for all $\sigma \in G$, we have by the multiplicativity if the representation $T_0 : \mathcal{K}_0^*\mathcal{K}_0 \to B(L_0)$, that for all $\sigma_1, \sigma_2 \in G$, $\Gamma\sigma_1\Gamma = \cup s_i\sigma_1\Gamma$ that

$$\sum_{s_i} T_0(s_i\sigma_1\Gamma)T_0(\Gamma\sigma_2) = T(\Gamma\sigma_1\Gamma)T(\Gamma\sigma_2).$$

5. CHAPTER 5

In this section we find an explicit formula for the absolute value of the matrix coefficients t_{13} of π_{13}, the 13-th element in the discrete series of $PSL_2(\mathbb{R})$, evaluated on a specific choice of Γ - wandering vector. In particular, this proves that, for these representation we may arrange so that $T^{\Gamma\sigma\Gamma}$ which a priori belongs to $l^2(\Gamma\sigma\Gamma) \cap \mathcal{L}(G)$ also belongs to the domain of ε. In fact, the coefficients $|t_{13}(\theta)|^2$ are $\mu_0(\theta F \cap F)$ where F is a fundamental domain in the upper half plane $\mathbb{H}$ acted by Γ (the choice of the fundamental domain corresponds to a choice of the Γ - wandering vector in H_{13}, which is unique up to a unitary in $\{\pi_{13}(\Gamma)\}' \cong R(\Gamma) \cong \mathcal{L}(\Gamma)$). Here μ_0 is the canonical $PSL_2(R)$ - invariant measure on the upperhalf plane $\mathbb{H}$.

First, we note the properties of the positive definite function we are looking for:

Proposition 21. *Let $\Gamma \subseteq G$ be a discrete group and let π be a representation of G, on a Hilbert space H, such that $\pi|_\Gamma$ is unitarely equivalent to the left regular representation of Γ (Thus $H \cong l^2(\Gamma)$) and there exists η in H such that $\pi(\gamma)\eta$ is orthogonal to η for all $\gamma \neq e$ and $\overline{Sp\{\pi(\gamma)\eta|\gamma \in \Gamma\}} = H$.*

Let $t(\theta) = \langle\pi(\theta)\eta, \eta\rangle$, $\theta \in G$. Then $t(\theta)$ is a positive definite function on G.

In the setting of Chapter 1 $t(\theta)$ is $\mathrm{Tr}(\pi(\theta)p_\eta)$, where p_η is the projection onto the 1-dimensional space generated by η. Then t has the following properties:

1) $t|_\Gamma(\gamma) = \delta_{\gamma,e}$, $\gamma \in \Gamma$, where e is the identity element of G and δ is the Kronecker symbol.

2) t is a positive definite function on G.

3) $\sum_{\gamma\in\Gamma}|t(\gamma g)|^2 = 1$.

In addition $t^{\Gamma\sigma\Gamma} = \sum_{\theta\in\Gamma\sigma\Gamma} t(\theta)\rho(\theta)$ *is a representation of* $\mathcal{H}$ *into* $R(G)$ *(this was proved in Chapter 1).*

Note that if π *extends to a larger, continuous group* $\overline{G}$ *which contains* G, *then property (3) holds on* $\overline{G}$ *also.*

Proof. The only fact that wasn't observed in Chapter 1 is property (3). But this follows from the fact that $\pi(g)\eta$ belongs to $\overline{S_p\{\pi(\gamma)\eta|\gamma\in\Gamma\}}$ and because $\pi(\gamma)\eta$, $\gamma\in\Gamma$, is an orthonormal basis.

□

Remark 22. In the previous setting, if $\varphi_0(g) = |t(g)|^2 \geq 0$, $g\in\overline{G}$ then φ_0 has exactly the behaviour of a state of the form $\mu(gF\cap F)$, $g\in\overline{G}$, where $\overline{G}$ acts on a measure space (X,μ), by measure preserving transformations. Then the properties (1), (3) correspond to the fact that F is a fundamental domain (and implicitly that $G(F) = \Gamma(F)$).

Note that the vector η is not unique, in fact all others vectors with similar properties to η, are of the form $u\eta$ where u is a unitary in $\{\pi(\Gamma)\}'' \cong R(\Gamma)$.

Thus the other equivalent forms of the representation of the Hecke algebra $\mathcal{H}$ given by $\Gamma\sigma\Gamma\to t^{\Gamma\sigma\Gamma}$ are of the form $\Gamma\sigma\Gamma\to u^* t^{\Gamma\sigma\Gamma}u$, $\sigma\in G$ for a unitaries u in $\mathcal{U}(R(\Gamma))$.

In the rest of the section $G = PGL_2(\mathbb{Z}[\frac{1}{p}])$, with p a prime number (or $G = PGL_2(\mathbb{Q})$), $\Gamma = PSL_2(\mathbb{Z})$ and $\overline{G} = PSL_2(\mathbb{R})$. The representation π will be the 13-th element in the discrete series of $PSL(2,\mathbb{R})$ (a projective unitary representation). The representation π_{13} acts on the Bargman Hilbert space H_{13} (see [Be], [Ra] for definitions). Recall $H_{13} = H^2(\mathbb{H}, (Imz)^{13-2}d\overline{z}dz)$.

We identify $H_{13}\otimes\overline{H}_{13}$ with $\mathcal{C}_2(H_{13})$. A typical element of $\mathcal{C}_2(H_{13})$ is the Toeplitz operator T_f^{13}, which in the sequel we denote by T_f simply, where f is a function of compact support in $\mathbb{H}$.

Recall that by (Be]), there exists a bounded operator $B\geq 0$, the Berezin's transform operator, with zero kernel, commuting with $\pi(g)$, $g\in\overline{G}$.

B is a function of the invariant Laplacian $\dfrac{\partial^2}{\partial z\partial\overline{z}}(Imz)^{-2}$ such that

$$\mathrm{Tr}(T_f T_g^*) = \langle B^{-1}f, g\rangle_{L^2(\mathbb{H},\mu_0)}.$$

Here μ_0 is the standard G invariant measure on $\mathbb{H}$.

We are looking to identify the vector η (or rather) $\eta \otimes \overline{\eta}$ in $H_{13} \otimes \overline{H}_{13}$ such that $|t(\theta)|^2 = \langle (\pi_{13} \otimes \overline{\pi}_{13})\eta \otimes \overline{\eta}, \eta \otimes \overline{\eta} \rangle$.

In the identification $H_{13} \otimes \overline{H}_{13}$ with $\mathcal{C}_2(H_{13})$ (the Scahtten von Neumann ideal of square summable operators on H_{13}) the representation π_{13} becomes $\mathrm{Ad}\pi_{13}$ on $\mathcal{C}_2(H_{13})$, and $\eta \otimes \overline{\eta}$ becomes the projection p_η onto $\mathbb{C}\eta$. Then $|t(\theta)|^2 = \mathrm{Tr}(p_\eta(\mathrm{Ad}\pi_{13}(\theta))(p_\eta))$ while $t(\theta) = \mathrm{Tr}(\pi_{13}(\theta)p_\eta)$, $\theta \in G$.

Let $\mathcal{C}_{13}$ be the closed selfdual cone in $\mathcal{C}_2(H_{13})$ generated by projections p_ζ, $\zeta \in H_{13}$, (that is $\mathcal{C}_{13}$ is the cone of positive elements). Then $\mathcal{C}_{13}$ is invariated by the representation $\mathrm{Ad}\pi_{13}$.

Remark 23. Because of the above formula for the scalar product in $\mathcal{C}_2(H_{13})$ of two Toeplitz operators, we have that we have a canonical unitary U between $L^2(\mathbb{H}, \mu_0)$, and $\mathcal{C}_2(H_{13})$, mapping functions f in $L^2(\mathbb{H}, \mu_0)$ into the Toeplitz operator $\{T_{B^{-1/2}f}$. Moreover the unitary U intertwines the corresponding representations of $PSL(2, R)$ and hence the selfdual closed cone $\mathcal{C}_{13}$ corresponds to the positive functions in $L^2(\mathbb{H}, \mu_0)$.

Proof. Indeed since $[\pi_{13}(g), B] = 0$ this is the only selfdual cone invariated by $\mathrm{Ad}\pi_{13}(g)$, $g \in G$.

□

Proposition 24. *Let F be a fundamental domain of Γ in $\mathbb{H}$.*

Let $f_0 = B^{1/2}\chi_F$. Then T_{f_0} is of the form p_η, where η is a Γ - wandering vector in H_{13}.

In particular the possible choices for the positive definite function φ_0 introduced in Remark 22 are

$$\varphi_0(g) = \mu_0(F \cap gF),\ g \in PSL_2(\mathbb{R}).$$

Proof. Indeed

$$\mathrm{Tr}(\pi_{13}(\gamma)T_{f_0}\pi_{13}(\gamma)^{-1}T_{f_0}) = \mathrm{Tr}(T_{B^{1/2}\chi_{\gamma F}}T_{B^{1/2}\chi_F}) =$$
$$= \langle B^{-1}B^{1/2}\chi_{\gamma F}, B^{1/2}\chi_F \rangle = 0$$

for $\gamma \in \Gamma \setminus \{e\}$.

Moreover $\sum\limits_{\gamma} \pi(\gamma)T_{f_0}\pi(\gamma)^{-1} = \sum T_{B^{-1/2}\chi_{\gamma F}}$ which is a multiple of the identity.

Thus $\mathrm{Tr}(T_{f_0})$ computes the trace either on $\{\pi_{13}(\Gamma)\}''$ and on $\{\pi_{13}(\Gamma)\}'$. Consequently it since T_{f_0} is positive (by the previous remark), it follows that the operators $\pi_{13}(\gamma)T_{f_0}\pi_{13}(\gamma)^{-1}, \gamma \in \Gamma$ have orthogonal ranges. Since we

have the Muray von Neumann dimension $\dim_{\Gamma H_{13}}$ is equal to 1, it follows that the operator $\mathrm{Tr}(T_{f_0})$ is of the form p_η, η a Γ-wandering cyclic vector in H_{13}. $\square$

We conjecture that in general, if G is a semisimple Lie group, and Γ a lattice in G, the convex set $K_{G,\Gamma}$ of continuous positive definite functions ϕ on G with the following the properties

1) $\phi(g) \geq 0$, for all g in G and ϕ is positive definite,
2) $\sum_{\gamma\in\Gamma} \phi(\gamma g) = 1$, all g in G,
3) $\phi(\gamma) = \delta_{\gamma,e}$, for $\gamma \in \Gamma$,

has extremal points Ext $K_{G,\Gamma}$, the positive definite functions of the form $\phi_F(g) = \mu_G(gF \cap F)$, where F is a fundamental domain for the action of Γ on G and μ_G is the left Haar measure on G, normalized so that $\mu_G(F) = 1$.

If G admits a unitary (projective) discrete series representation π on a Hilbert space H such that $\pi|_\Gamma$ is unitarily equivalent to the left regular representation of Γ, then we conjecture that Ext $K_{G,\Gamma}$ coincides with the positive definite function of the form

$$g \to |\langle \pi(g)\eta, \eta\rangle|^2, g \in G$$

where η runs over vectors in H that are trace vectors for $\pi(\Gamma)$.

Corollary 25. *In the previous context, we have that for $\theta \in G$*

$$\varphi_0(\theta) = |t(\theta)|^2 = \mathrm{Tr}(T_{f_0}\pi_{13}(\theta)T_{f_0}\pi_{13}(\theta)^{-1}) =$$
$$= \langle B^{-1}B^{1/2}\chi_F, B^{1/2}\chi_{\theta F}\rangle_{L^2(\mathbb{H},\mu_0)} = \mu_0(\theta F \cap F).$$

Also $t(\theta) = \mathrm{tr}(\pi_{13}(\theta)T_{B^{1/2}\chi_F})$, $\theta \in G$.

We conjecture that $T_{B^{1/2}\chi_F}$ is the element $\zeta^{-1/2}T_{\chi_F}\zeta^{-1/2}$ where ζ is the square root of the positive definite function on Γ defined by the formula $\gamma \to \mathrm{Tr}(\pi_{13}(\gamma)\chi_F), \gamma \in \Gamma$ (that is the element $E_{R(G)\otimes 1}(p)$ from Proposition 11). By $\zeta^{-1/2}$ we understand in fact $\pi_{13}(\zeta^{-1/3})$, its image through the representation π_{13}.

Corollary 26. *With the above notations, if F is a fundamental domain for Γ and $\phi_0(\theta) = \mu_0(\theta F \cap F), \theta \in G$, then ϕ_0 defines a positive definite function ψ_0 on $\mathcal{K}_0^*\mathcal{K}_0$, by the formula*

$$\psi_0(\sigma_1\Gamma, \Gamma\sigma_2) = \sum_{\gamma\in\Gamma} \mu_0(\sigma_1\gamma\sigma_2 F \cap F) = \mu_0(\sigma_1\Gamma\sigma_2 F \cap F), \sigma_1, \sigma_2 \in G.$$

Here positivity means that for all $k_1, k_2, ..., k_n \in \mathcal{K}_0$ the matrix

$$(\psi_0(k_i^*, k_j))_{i,j=1,2,...,n}$$

is positive definite. Since $\psi_0|_{\mathcal{H}_0}$ is the character corresponding to the identity on $\mathcal{H}_0$, then ψ_0 is a positive extension of this character to $\mathcal{K}_0^*\mathcal{K}_0$ in the sense of Chapter 4.

Note that $\psi_0(\sigma_1\Gamma, \Gamma\sigma_2)$ may be interpreted as the matrix coefficient of the representation of σ_1 on $L^2(\mathbb{H}, \mu_0)$ at the Γ equivariant vectors $\chi_{\sigma_2\Gamma F}$ and $\chi_{\Gamma F} = \chi_{\mathbb{H}}$. Note that χ_F is a cyclic vector for the representation of G on $L^2(\mathbb{H}, \mu_0)$, so in principle ϕ_0 contains all the information about the representation, including action on Γ invariant vectors.

Proof. Proof because of Proposition 25, we have, with the notations from ([Ra]), that for all $\sigma_1, \sigma_2 \in G$,

$$\psi_0(\sigma_1\Gamma, \Gamma\sigma_2) = \sum_{\theta\in\sigma_1\Gamma\sigma_2} |t(\theta)|^2 = \tau((t^{\sigma_1\Gamma}t^{\Gamma\sigma_2})^*(t^{\sigma_1\Gamma}t^{\Gamma\sigma_2}).$$

Since τ is the trace on the group algebra of G, this is further equal to

$$\tau(t^{\sigma_1^{-1}\Gamma\sigma_1}t^{\sigma_2^{-1}\Gamma\sigma_2}).$$

This implies the positivity of ψ_0

□

Note that there exists a construction of Merel [Me] (we are indebted to Alex Popa for bringing to our attention this paper), of a finitely supported element X in $\mathbb{C}(\Gamma\sigma_p\Gamma)$, such that $\Gamma\sigma_p\Gamma \to X$ extends to a representation of the Hecke algebra (p is a prime number).

Unfortunately, for the element X constructed in the paper ([Me]), the property that the representation of the Hecke algebra may be extended to $\mathcal{K}_0^*\mathcal{K}_0$ does not hold true. The extension property is, as proved in ([Ra]), the main ingredient that allows to represent Hecke operators on Maass forms, by using the above representation of the Hecke algebra (see also the construction in Theorem 8).

The following lemma gives a characterization of the elements X in $l^2(\Gamma\sigma_p\Gamma) \cap \mathcal{L}(G)$, that generate a representation of $\mathcal{K}^*\mathcal{K}$ into $\mathcal{L}(G)$, coming from a representation π of G, which has the property that $\pi|_\Gamma$ is unitarely equivalent to the left regular representation of Γ.

Proposition 27. *Let X be a selfadjoint element in $l^2(\Gamma\sigma_p\Gamma)$. Let χ_1 be the radial element in the group algebra of $F_{\frac{p+1}{2}}$ (the free group with $\dfrac{p+1}{2}$ generators) (thus $\chi_n = \sum_w w$, where w runs over words of length n, $n \in \mathbb{N}$).*

Let τ_p be the trace on $\mathcal{L}(F_{\frac{p+1}{2}})$.

(1) Then $(\Gamma\sigma_p\Gamma)^n \to X^n$ *extends to a representation of the Hecke algebra if and only if* $\tau_{\mathcal{L}(G)}(X^n) = \tau_p(\chi_1^n)$.

(2) Let X_n *be the projection of* X^n *onto* $l^2(\Gamma\sigma_{p^n}\Gamma)$. *Let* $X_{\Gamma\sigma_{p^e}s}$ *(respectively* $X_{s\sigma_{p^e}\Gamma}$*), for* $s \in G$, $e \in \mathbb{N}$, *be the projection of* X_e *onto* $l^2(\Gamma\sigma_{p^e}s)$ *(respectively* $l^2(s\sigma_{p^e}\Gamma)$*).*

Then X *comes from a representation* π *of* G *on* H *(with* $\dim_\Gamma H = 1$*) as in Chapter 1 (or [Ra]) if and only if* $(X_{\Gamma\sigma})^* = (X_{\sigma\Gamma})$, $\sigma \in G$ *and* $[\sigma_1\Gamma][\Gamma\sigma_2] \to X_{\sigma_1\Gamma}X_{\Gamma\sigma_2}$, $\sigma_1, \sigma_2 \in G$ *extends to a representation of* $\mathcal{K}_0^*\mathcal{K}_0$, *that is* $X_{\sigma_1\Gamma}X_{\Gamma\sigma_2} = X_{\sigma_1\Gamma\sigma_2}$, *where* $X_{\sigma_1\Gamma\sigma_2}$ *is the projection onto* $l^2(\sigma_1\Gamma\sigma_2)$ *of* $X_{\sigma_1\Gamma}X_{\Gamma\sigma_2\Gamma}$ *or of* $X_{\Gamma\sigma_1\Gamma}X_{\Gamma\sigma_2}$ *(Equivalently* $X_{\sigma_1\Gamma\sigma_2}$ *is the projection onto* $l^2(\sigma_1\Gamma\sigma_2)$ *of* $\sum\limits_{e\in\mathcal{I}} X_e$, *where* $\bigcup\limits_{e\in\mathcal{I}}[\Gamma\sigma_{p^e}\Gamma]$ *covers* $\sigma_1\Gamma\sigma_2$*) (this should be true for all* $\sigma_1, \sigma_2 \in G$*).*

If these conditions are verified, there $\Psi(a) = E_{\mathcal{L}(\Gamma)}^{\mathcal{L}(G)}(XaX)$, $a \in \mathcal{L}(\Gamma)$ *defines a representation of the Hecke algebra (into the algebra generated by completely positive maps on* $\mathcal{L}(\Gamma)$*).*

(3) The condition in (2) is verified if the following weaker condition holds true

$$X_{\Gamma\sigma_1\Gamma}X_{\Gamma\sigma_2} = \sum_{r_j} X_{\Gamma\sigma_1 r_j\sigma_2}$$

where $\Gamma\sigma_1\Gamma$ *is the disjoint union of* $\Gamma\sigma_1 r_j$. *This property should be verified for all* σ_1, σ_2. *Thus to verify condition (2) it is sufficient to verify that* X *(and* $X_n,\ n \geq 1$*) act on* $Sp\{X_{\Gamma\sigma} \mid \sigma \in G\}$ *exactly as* $[\Gamma\sigma_{p^n}\Gamma]$ *acts on* $l^2(\Gamma/G)$.

Remark 28. The condition (2) automaticaly implies that $(X_{\Gamma\sigma})^*X_{\Gamma\sigma}$ is equal to $X_{\sigma\Gamma\sigma}$, where only component in Γ is τ_e, and thus $\tau_{\mathcal{L}(G)}((X_{\Gamma\sigma})^*X_{\Gamma\sigma})$ is 1, thus $\|X_{\Gamma\sigma}\|^2_{2,\mathcal{L}(G)} = 1$.

Proof. To prove (1) observe that the fact that $\tau_{\mathcal{L}(G)}(X^n) = \tau(\chi_1^n)$, for all $n \in \mathbb{N}$, is equivalent to the fact X^n decomposes as a linear space of $\{X_k \mid k \leq n\}$ exactly as χ_1^n decomposes as a linear space of $\{\chi_k \mid k \leq n\}$ (i.e. with the same coefficients).

Hence $\chi_n \to X_n$ is a $*$ isomorphism but this implies that $\Gamma\sigma_{p^n}\Gamma \to X_n$ is a $*$ isomorphism, which because of orthogonality extends to the weak closure of the algebras involved.

To prove (2), we only have to prove the converse (since the direct implication was prove in [Ra], and reproved in Chapter 1). If condition (2) is verified one defines a representation π of G on $l^2(\Gamma)$ simply by defining $\pi(\sigma)1$ (where 1 is the identity of the group Γ, viewed as an element in $l^2(\Gamma)$) to be

$\sigma(t^{\Gamma\sigma})^*$. This then, by requiring that $\pi|_\Gamma = \lambda_\Gamma$ defines the entire representation π.

The fact that (3) implies (2) is a simple consequence of supports computations. Indeed, if we take $\sigma_1, \sigma_2 \in G$ then if $\Gamma\sigma_1\Gamma = \cup s_i\sigma_1\Gamma$ (as a disjoint union), then the cosets $(s_i\sigma_1)\Gamma\sigma_2$ are disjoint and their union is the union of the cosets in $[\Gamma\sigma_1\Gamma][\Gamma\sigma_2]$.

□

REFERENCES

[Be] Berezin, F. A. Quantization. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 1116–1175.

[BC] Bost, J.-B.; Connes, A., Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory. Selecta Math. (N.S.) **1** (1995), no. 3, 411–457.

[Co] , Connes, A., A unitary invariant in Riemannian geometry. (English summary) Int. J. Geom. Methods Mod. Phys. 5 (2008), no. 8, 12151242.

[BH] Bridson, Martin R.; de la Harpe, Pierre Mapping class groups and outer automorphism groups of free groups are C^*-simple. J. Funct. Anal. 212 (2004), no. 1, 195–205.

[GHJ] Goodman, Frederick M.; de la Harpe, Pierre; Jones, Vaughan F.R., Coxeter graphs and towers of algebras. Mathematical Sciences Research Institute Publications, **14**. Springer-Verlag, New York, 1989, x+288 pp.

[Ke] A.S. Kechis, Global aspects of ergodic group actions, Amer. Math. Soc., Mathematical Survey and Monographs, vol. 160, 2010.

[Mvn] Murray, F. J.; Von Neumann, J. On rings of operators. Ann. of Math. (2) 37 (1936), no. 1, 116229.

[Pi] Pisier, Gilles Introduction to operator space theory. London Mathematical Society Lecture Note Series, 294. Cambridge University Press, Cambridge, 2003.

[Ra] F. Rădulescu, Type II_1 von Neumann representations for Hecke operators on Maass forms and Ramanujan-Petersson conjecture.

[Me] Merel, Loc Oprateurs de Hecke et sous-groupes de ?(2). J. Number Theory 50 (1995), no. 2, 345362.

[Ta] Takesaki, M. Theory of operator algebras. I. Reprint of the first (1979) edition. Encyclopaedia of Mathematical Sciences, 124. Operator Algebras and Noncommutative Geometry, 5. Springer-Verlag, Berlin, 2002. xx+415 pp

[Tz] Tzanev, Kroum, Hecke C^*-algebras and amenability. J. Operator Theory **50** (2003), no. 1.

[[Za] Zagier, Don Traces des oprateurs de Hecke. (French) Sminaire Delange-Pisot-Poitou, 17e anne: 1975/76. Thorie des nombres: Fasc. 2, Exp. No. 23, 12 pp. Secrtariat Math., Paris, 1977.